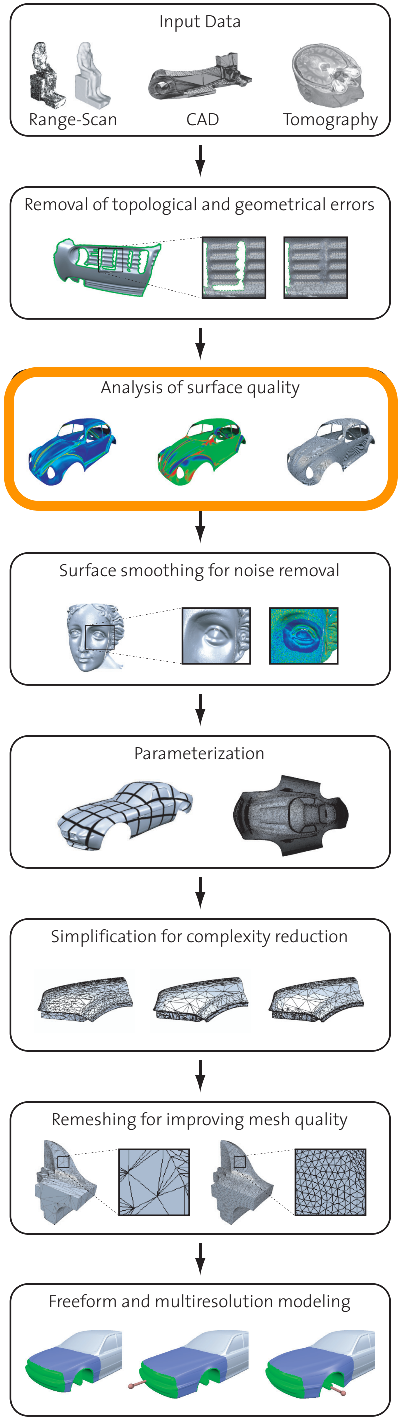

Differential Geometry

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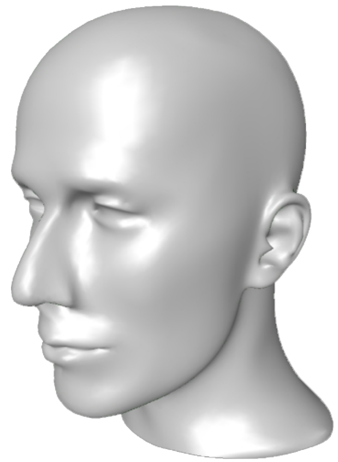
Outline

- Differential Geometry
  - curvature
  - fundamental forms
  - Laplace-Beltrami operator
- Discretization
- Visual Inspection of Mesh Quality

Differential Geometry

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \ (u,v) \in \mathbb{R}^2$$



- Normal vector

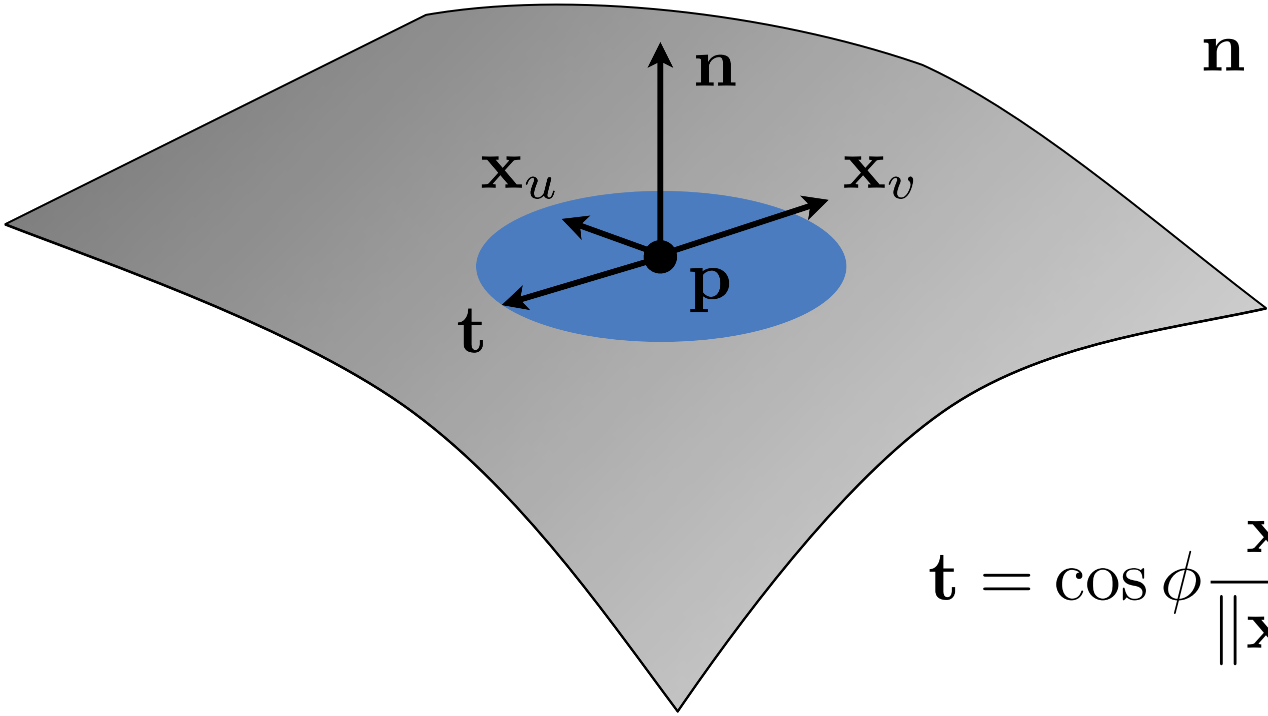
$$\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$$

  – assuming regular parameterization, i.e.

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Differential Geometry

- Normal Curvature

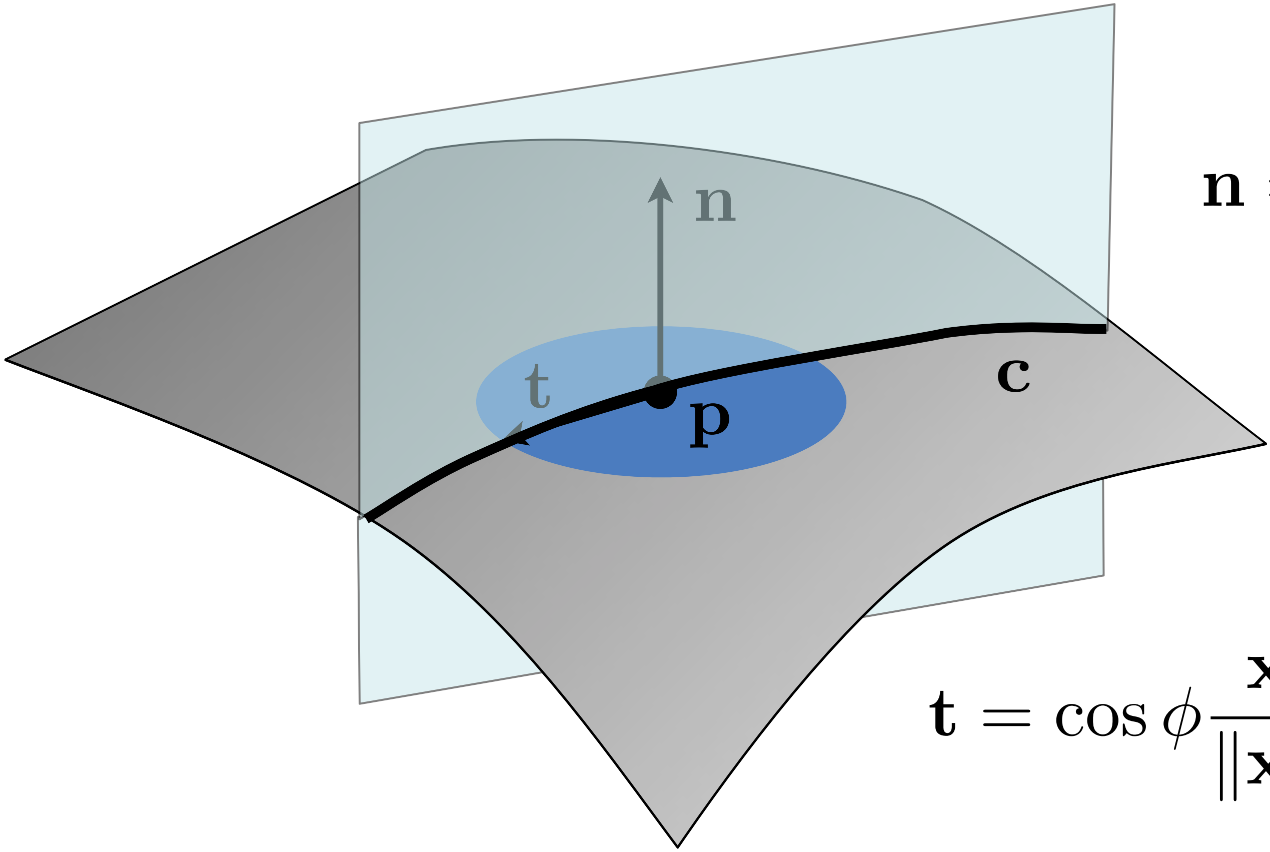


$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Differential Geometry

- Normal Curvature



$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

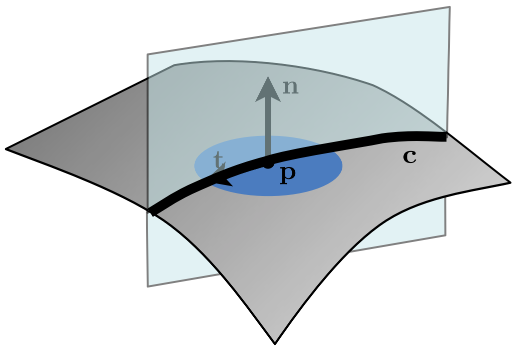
Differential Geometry

- Principal Curvatures
  - maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Euler Theorem: $\kappa_n(\bar{\mathbf{t}}) = \kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
- Mean Curvature $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\phi) d\phi$
- Gaussian Curvature $K = \kappa_1 \cdot \kappa_2$

Differential Geometry

- Normal curvature is defined as curvature of the normal curve $\mathbf{c} \in \mathbf{x}(u, v)$ at a point $\mathbf{p} \in \mathbf{c}$
- Can be expressed in terms of fundamental forms as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I}\, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$



$$\mathbf{t} = a\mathbf{x}_u + b\mathbf{x}_v$$

Differential Geometry

- First fundamental form

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

- Second fundamental form

$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

Differential Geometry

- $\mathbf{I}$ and $\mathbf{II}$ allow to measure
  - length, angles, area, curvature
  - arc element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

  - area element

$$dA = \sqrt{EG - F^2}dudv$$

Differential Geometry

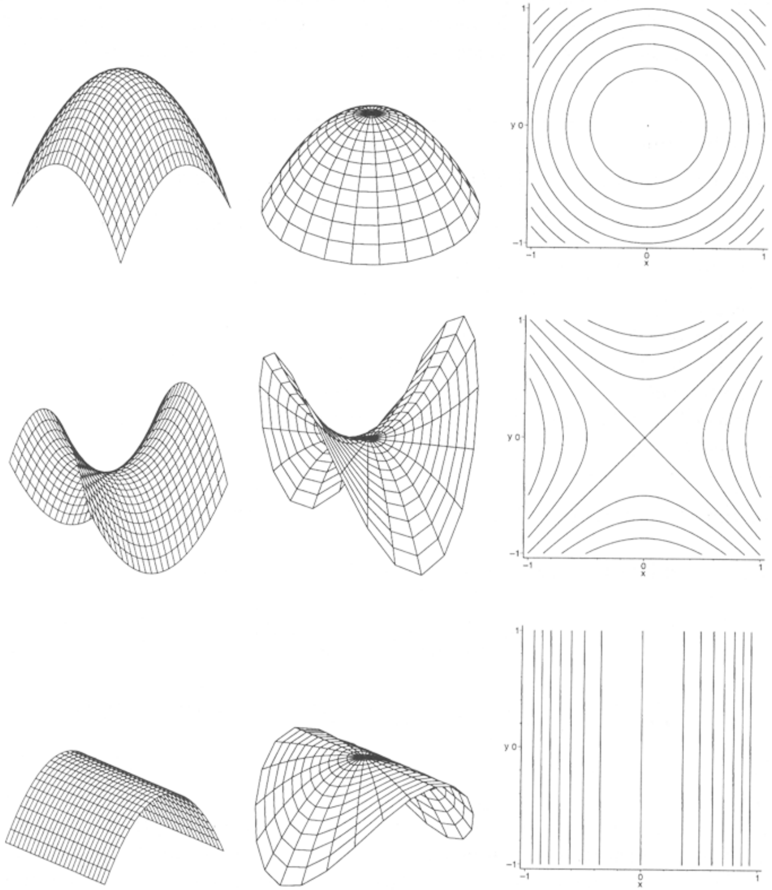
- Intrinsic geometry: Properties of the surface that only depend on the first fundamental form
  - length
  - angles
  - Gaussian curvature (Theorema Egregium)

$$K = \lim_{r \to 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

Differential Geometry

- A point $\mathbf{x}$ on the surface is called
  - *elliptic*, if $K > 0$
  - *parabolic*, if $K = 0$
  - *hyperbolic*, if $K < 0$
  - *umbilical*, if $\kappa_1 = \kappa_2$

- Developable surface ⇔ $K = 0$



Laplace Operator

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Laplace operator

gradient operator

2nd partial derivatives

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} f$$

function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

Laplace-Beltrami

gradient operator

mean curvature

coordinate function

divergence operator

surface normal

Outline

- Differential Geometry
  - curvature
  - fundamental forms
  - Laplace-Beltrami operator
- **Discretization**
- Visual Inspection of Mesh Quality

Discrete Differential Operators

- Assumption: Meshes are piecewise linear approximations of smooth surfaces
- Approach: Approximate differential properties at point x as spatial average over local mesh neighborhood $N(x)$, where typically
  - x = mesh vertex
  - $N(x)$ = n-ring neighborhood or local geodesic ball

Discrete Laplace-Beltrami

- Uniform discretization

$$\Delta_{uni} f(v) := \frac{1}{|\mathcal{N}_1(v)|} \sum_{v_i \in \mathcal{N}_1(v)} (f(v_i) - f(v))$$

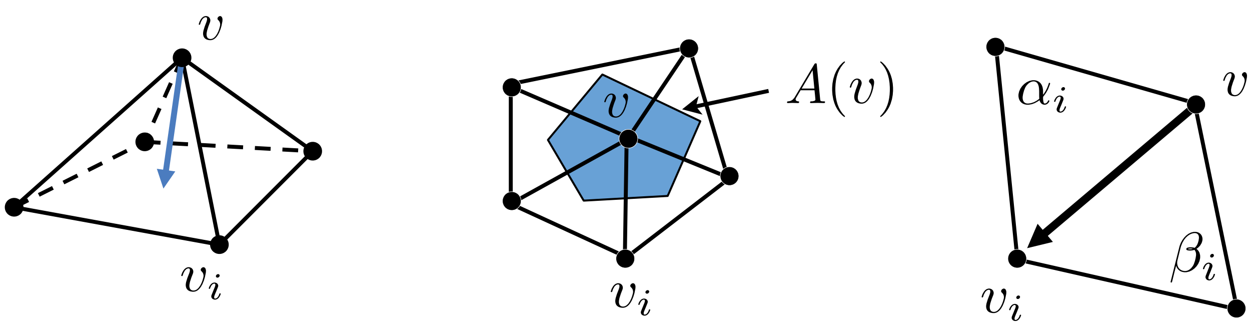
  – depends only on connectivity → simple and efficient

  – bad approximation for irregular triangulations

Discrete Laplace-Beltrami

- Cotangent formula

$$\Delta_{\mathcal{S}} f(v) := \frac{2}{A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) \left(f(v_i) - f(v) \right)$$



Discrete Laplace-Beltrami

- Cotangent formula

$$\Delta_{\mathcal{S}} f(v) := \frac{2}{A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) \left(f(v_i) - f(v) \right)$$

- Problems
  - negative weights
  - depends on triangulation

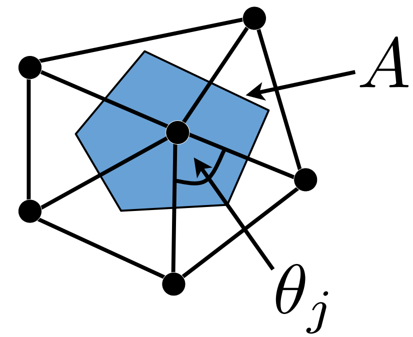
Discrete Curvatures

- Mean curvature

$$H = \|\Delta_{\mathcal{S}} \mathbf{x}\|$$

- Gaussian curvature

$$G = (2\pi - \sum_j \theta_j)/A$$



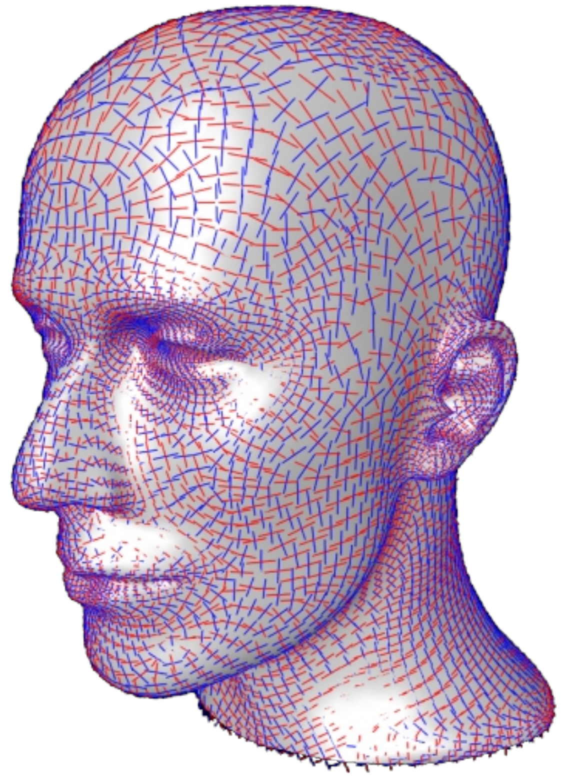
- Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - G} \qquad \kappa_2 = H - \sqrt{H^2 - G}$$

Links & Literature

- P. Alliez: *Estimating Curvature Tensors on Triangle Meshes* (source code)
  - http://www-sop.inria.fr/geometrica/team/Pierre.Alliez/demos/curvature/

- Wardetzky, Mathur, Kaelberer, Grinspun: *Discrete Laplace Operators: No free lunch*, SGP 2007



principal directions

Outline

- Differential Geometry
  - curvature
  - fundamental forms
  - Laplace-Beltrami operator
- Discretization
- **Visual Inspection of Mesh Quality**

Mesh Quality

- Smoothness
  - continuous differentiability of a surface (C^k)
- Fairness
  - aesthetic measure of "well-shapedness"
  - principle of simplest shape
  - fairness measures from physical models

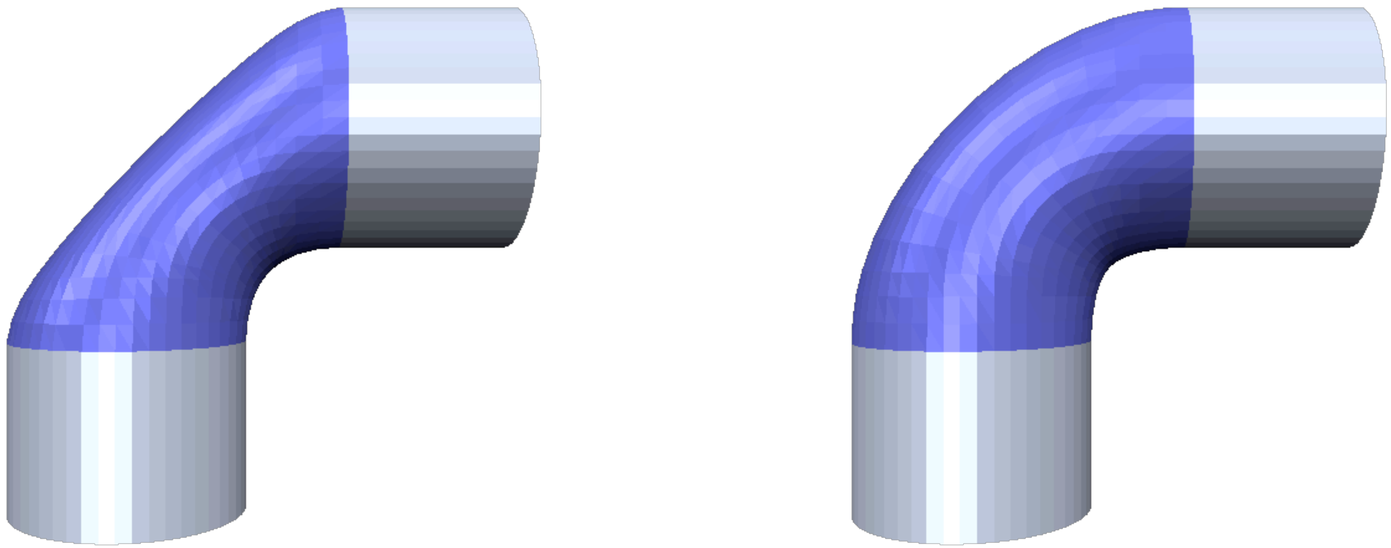
$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \, dA$$

strain energy

$$\int_{\mathcal{S}} \left(\frac{\partial \kappa_1}{\partial \mathbf{t}_1}\right)^2 + \left(\frac{\partial \kappa_2}{\partial \mathbf{t}_2}\right)^2 dA$$

variation of curvature

Mesh Quality



$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \, dA$$

strain energy

$$\int_{\mathcal{S}} \left(\frac{\partial \kappa_1}{\partial \mathbf{t}_1}\right)^2 + \left(\frac{\partial \kappa_2}{\partial \mathbf{t}_2}\right)^2 dA$$

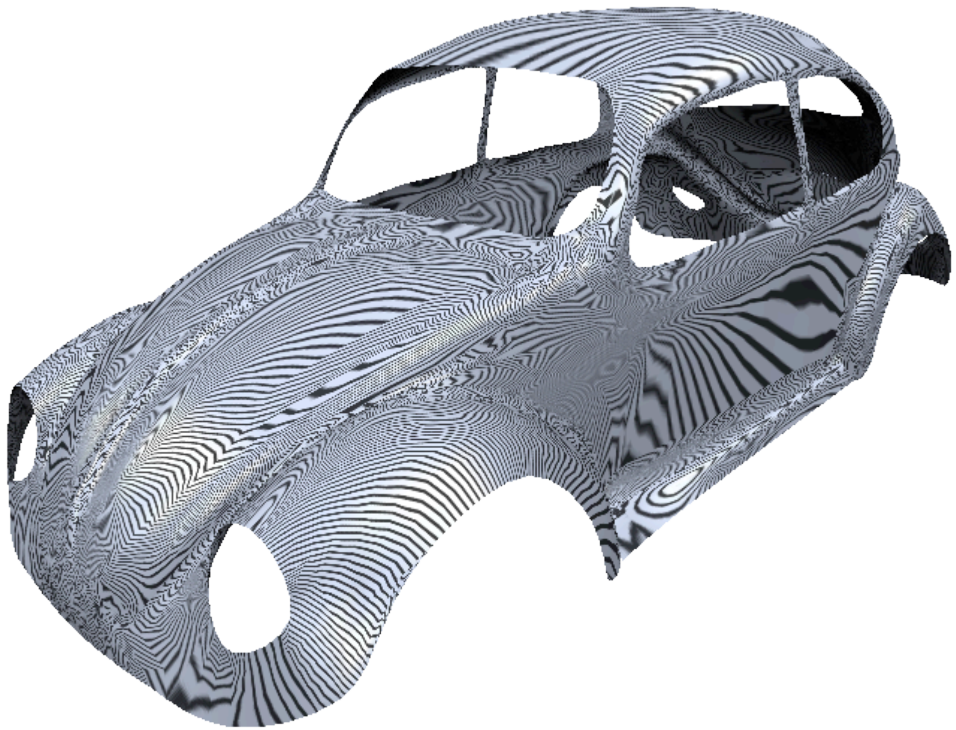
variation of curvature

Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading

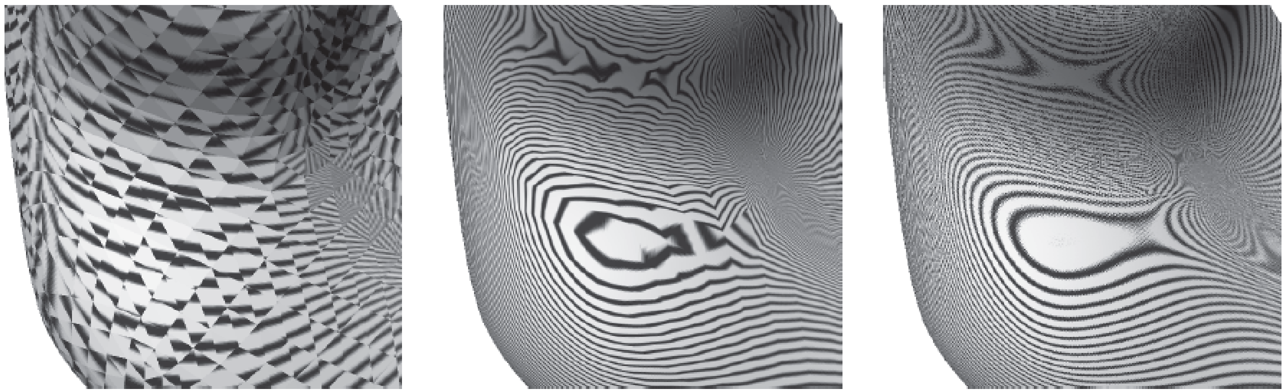
Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines



Mesh Quality

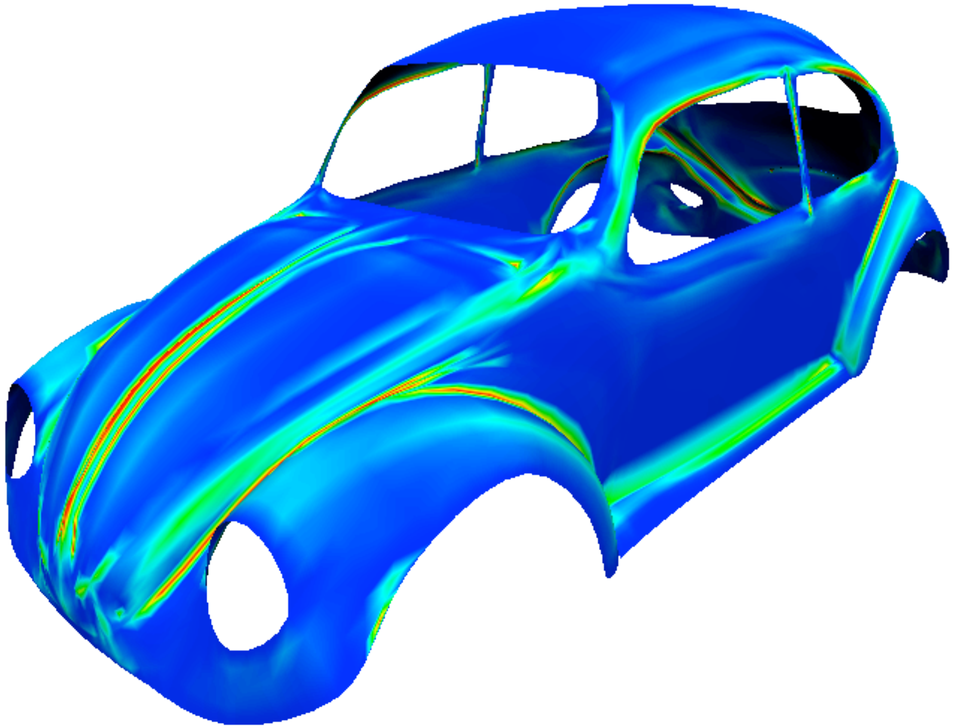
- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
    - differentiability one order lower than surface
    - can be efficiently computed using graphics hardware



C^0 C^1 C^2

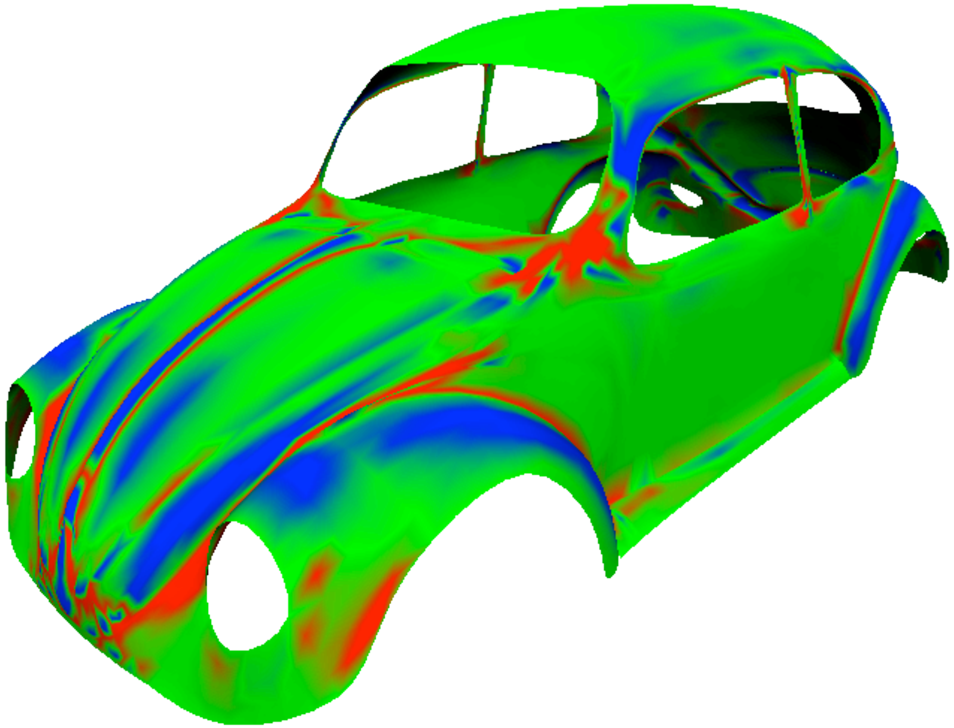
Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature



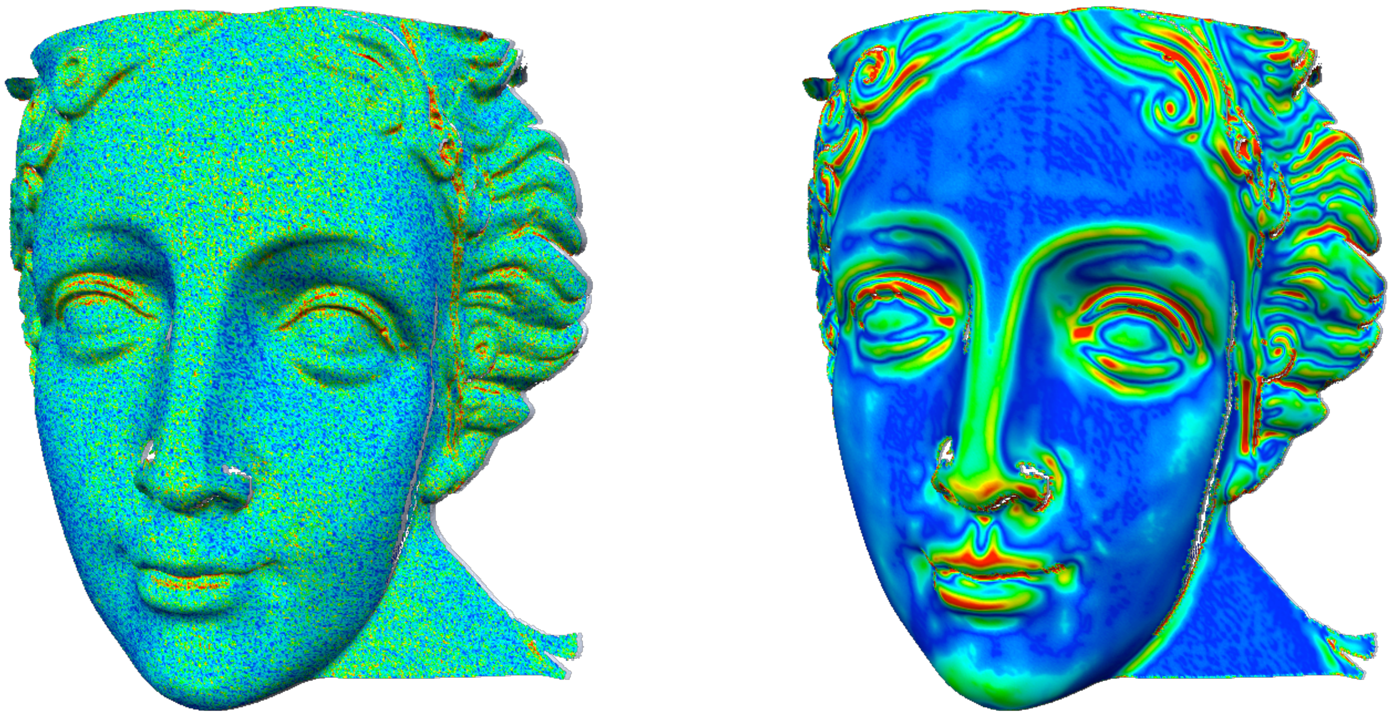
Mesh Quality

- Visual inspection of "sensitive" attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature
    - Gauss curvature



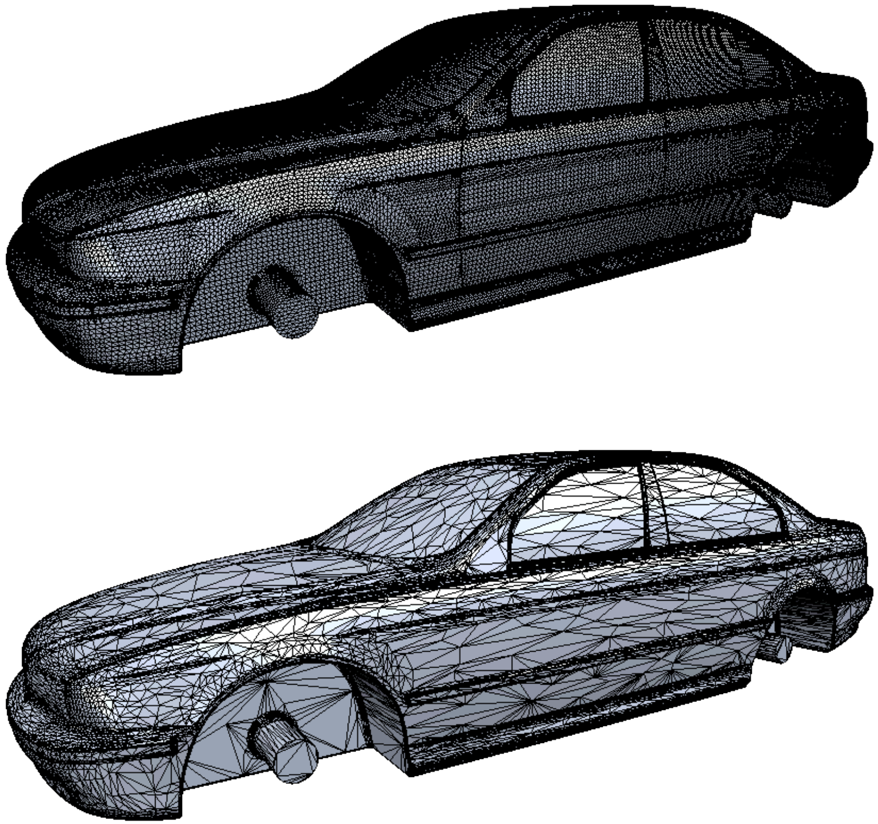
Mesh Quality Criteria

- Smoothness
  - Low geometric noise



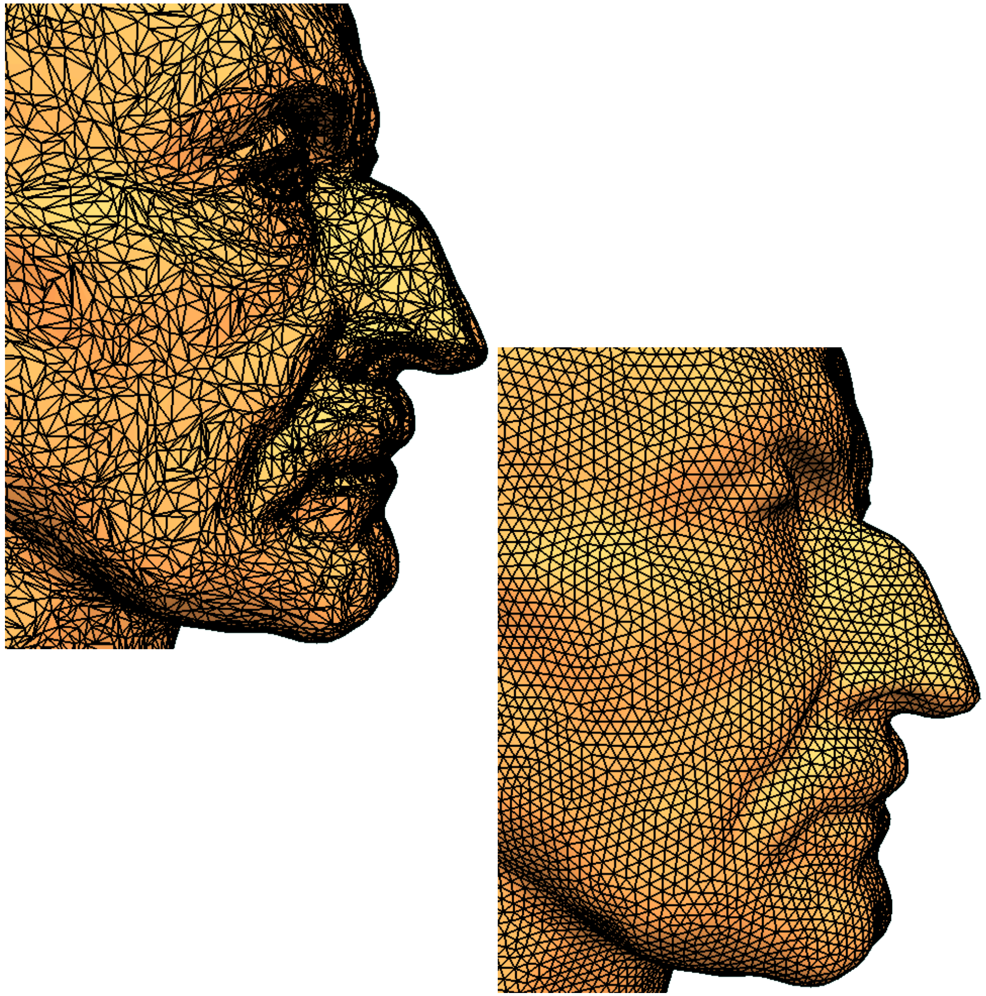
Mesh Quality Criteria

- Smoothness
  - Low geometric noise
- Adaptive tessellation
  - Low complexity



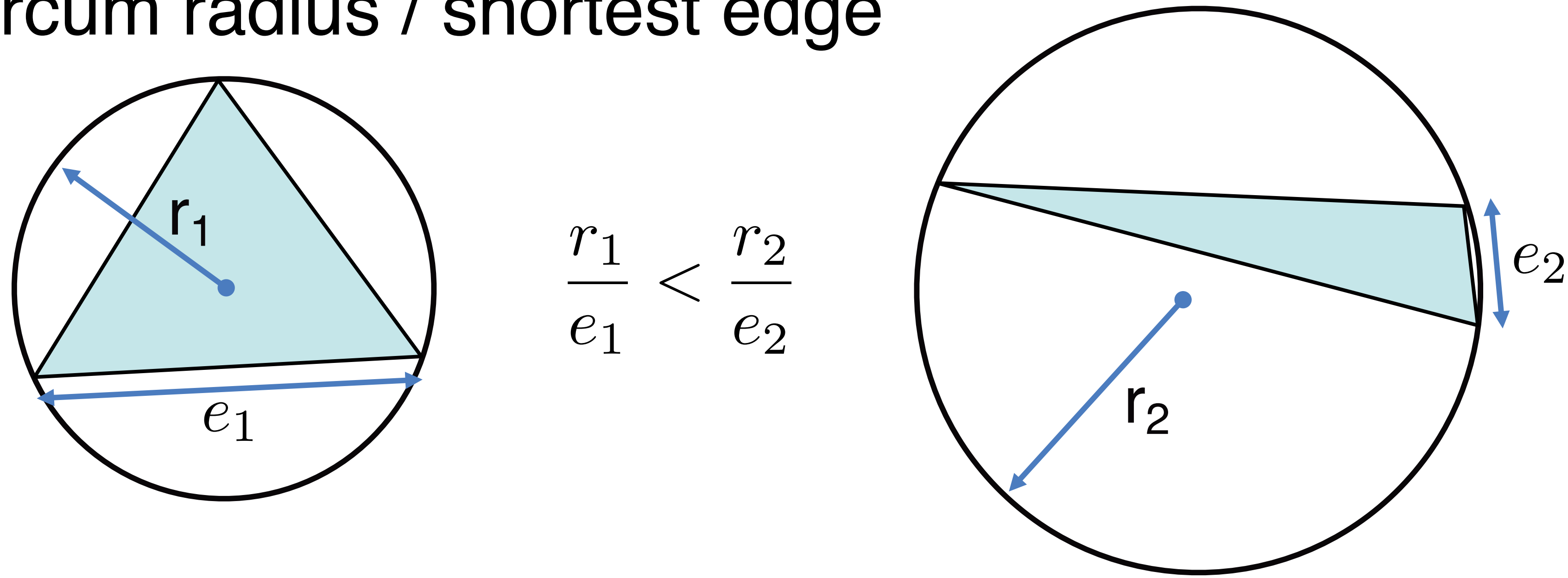
Mesh Quality Criteria

- Smoothness
  - Low geometric noise
- Adaptive tessellation
  - Low complexity
- Triangle shape
  - Numerical robustness



Triangle Shape Analysis

- Circum radius / shortest edge



$$\frac{r_1}{e_1} < \frac{r_2}{e_2}$$

- Needles and caps

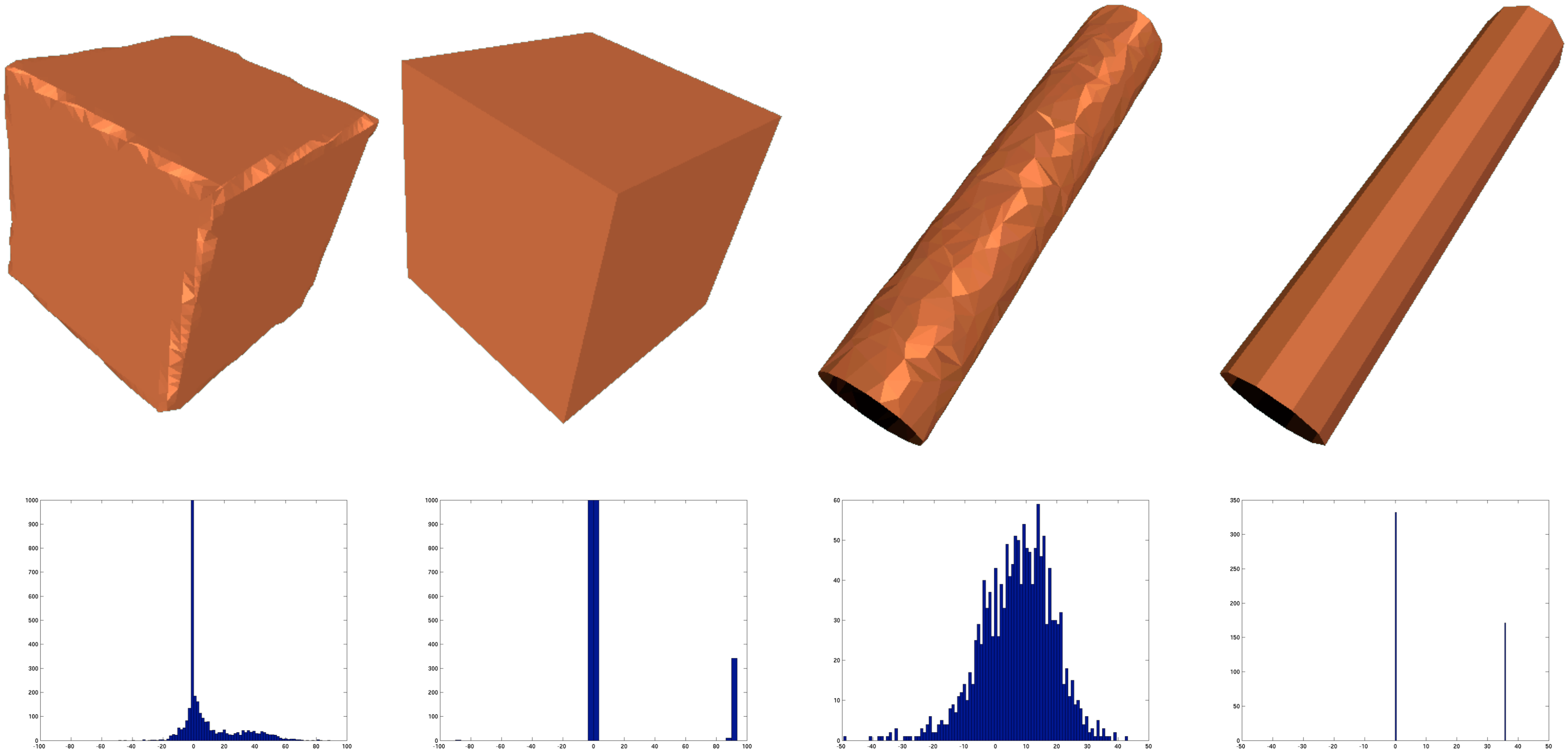
Needle

Cap

Mesh Quality Criteria

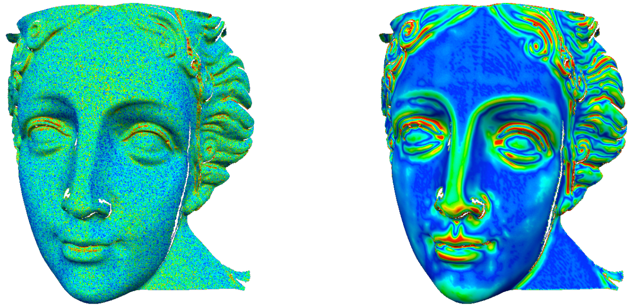
- Smoothness
  - Low geometric noise
- Adaptive tessellation
  - Low complexity
- Triangle shape
  - Numerical robustness
- Feature preservation
  - Low normal noise

Normal Noise Analysis

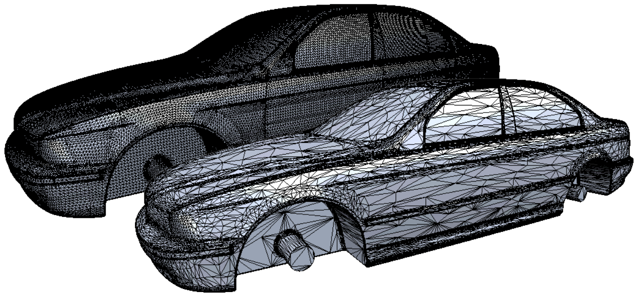


Mesh Optimization

- Smoothness
  - ➡ Mesh smoothing



- Adaptive tessellation
  - ➡ Mesh decimation



- Triangle shape
  - ➡ Repair, remeshing