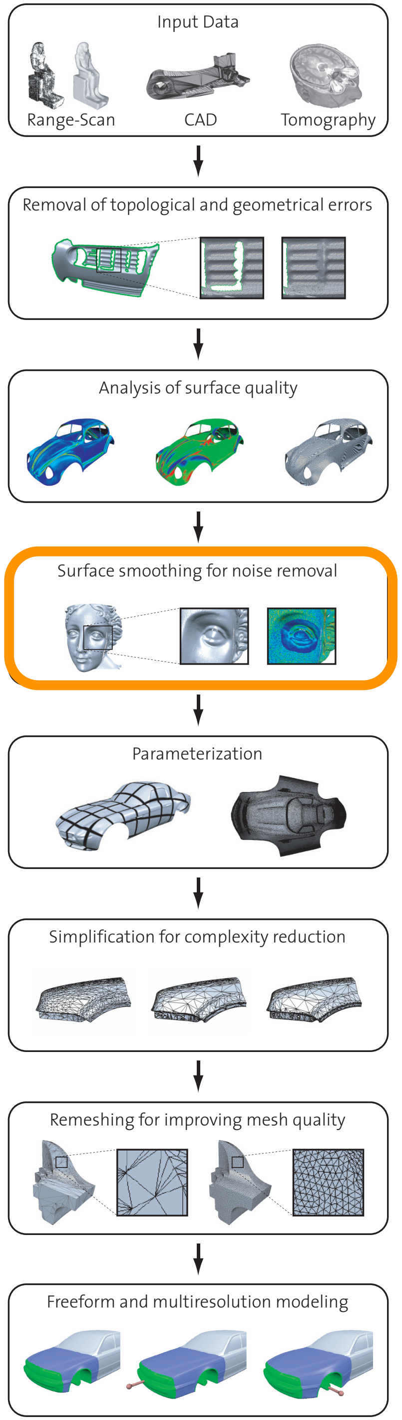


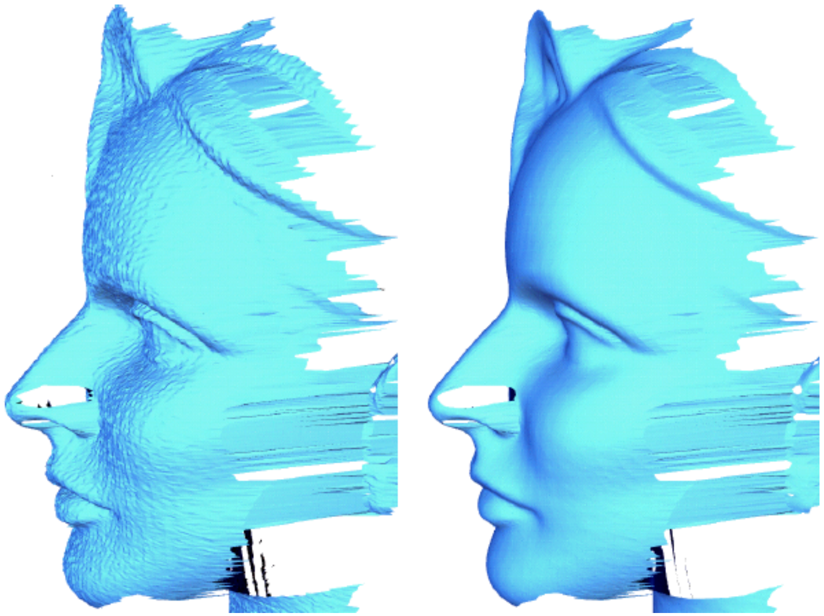
Mesh Smoothing

Mark Pauly

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Motivation

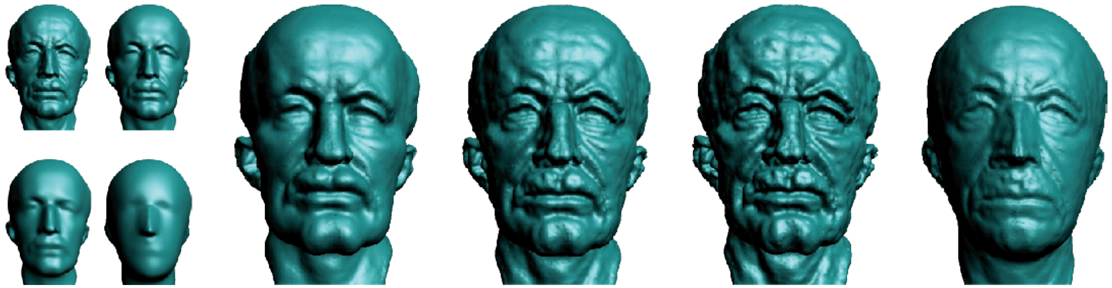
- Filter out high frequency components for noise removal



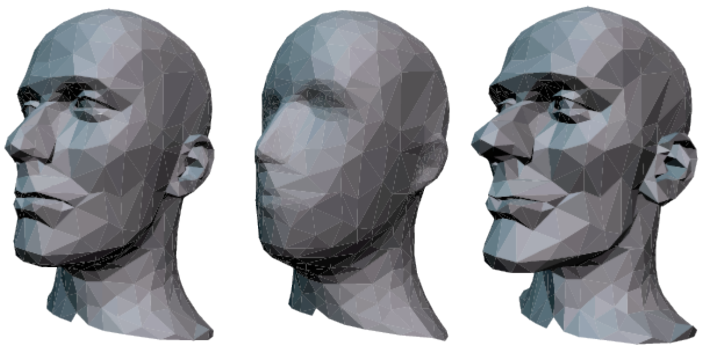
Desbrun, Meyer, Schroeder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 99

Motivation

- Advanced Filtering / Signal Processing



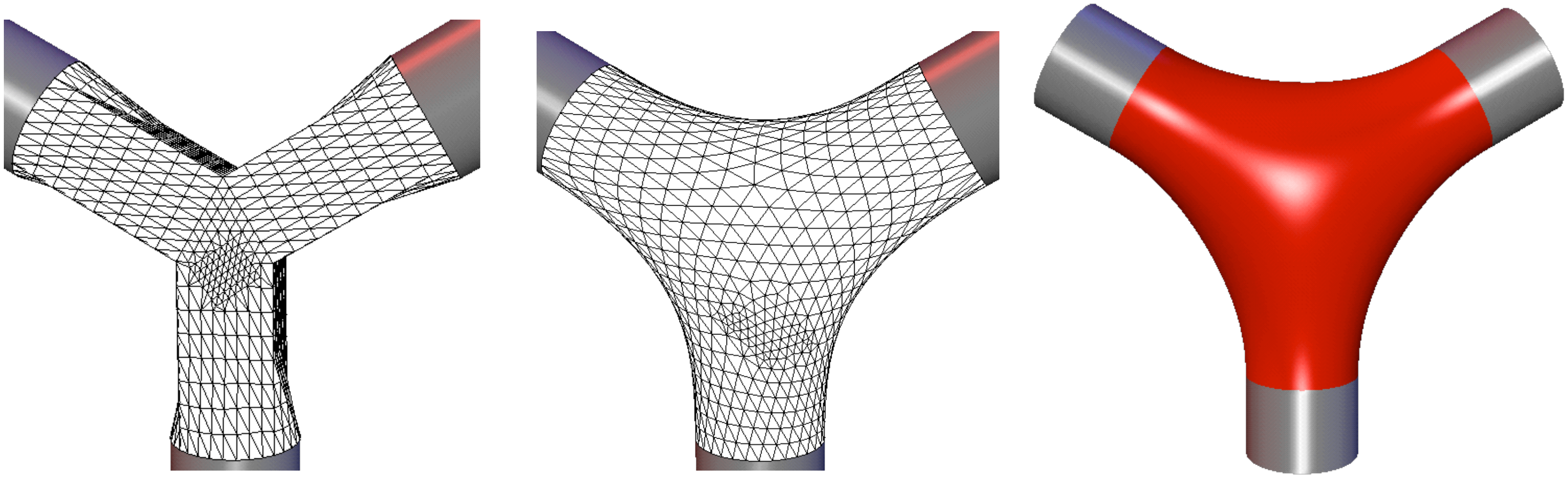
Pauly, Kobbelt, Gross: *Point-Based Multi-Scale Surface Representation*, ACM TOG 2006



Guskow, Sweldens, Schroeder: *Multiresolution Signal Processing for Meshes*, SIGGRAPH 99

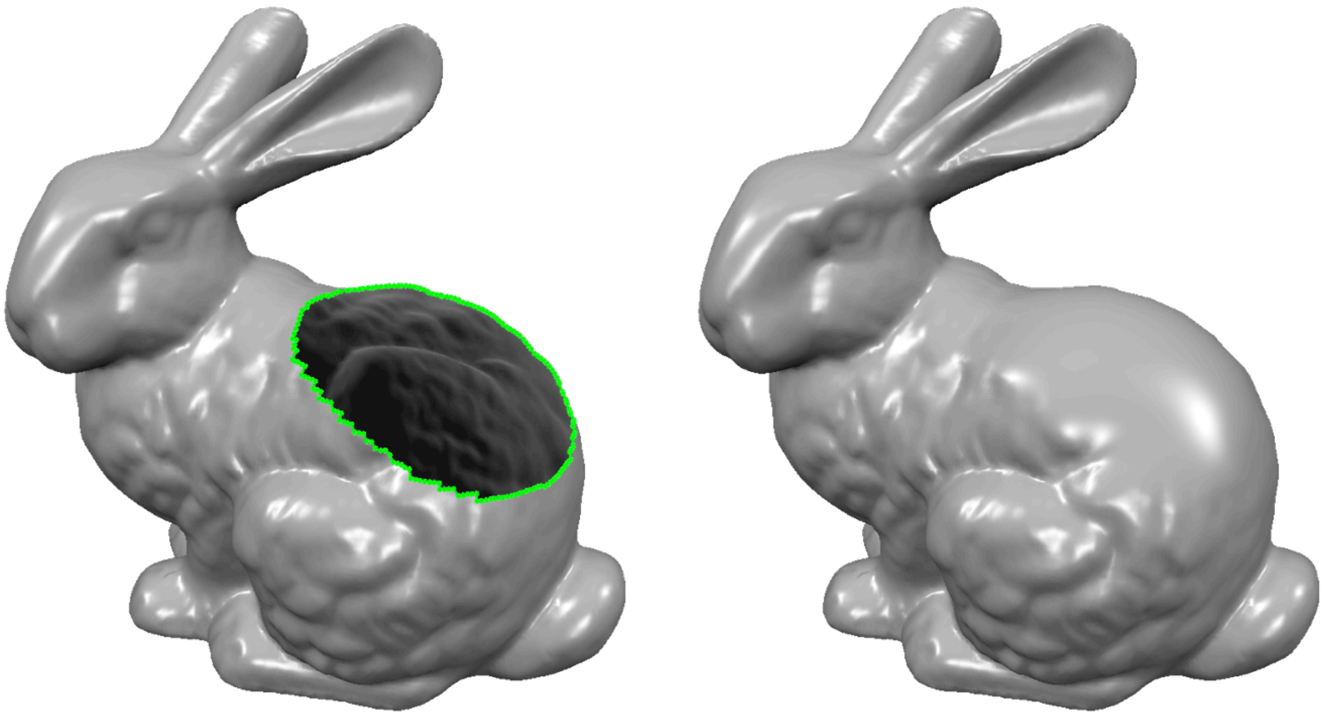
Motivation

- Fair Surface Design



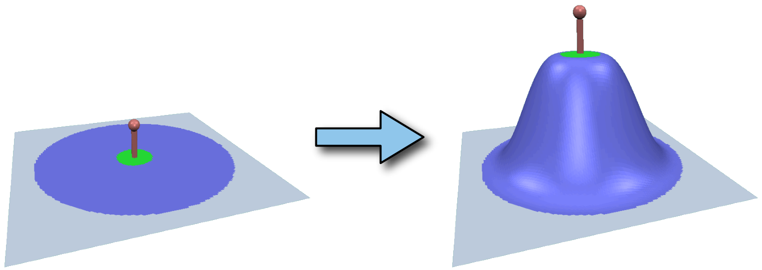
Motivation

- Hole-filling with energy-minimizing patches

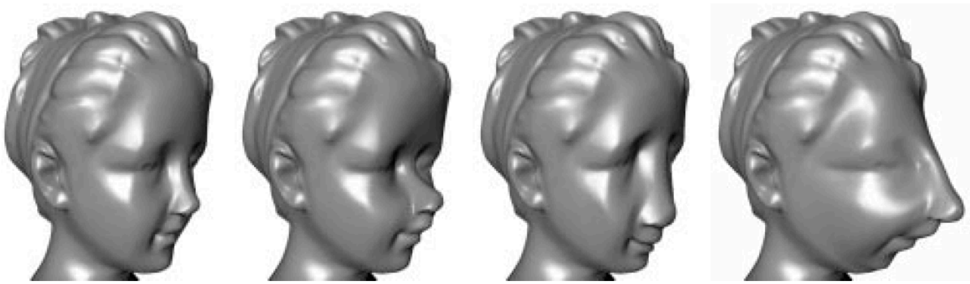


Motivation

- Mesh deformation



Botsch, Kobbelt: *An intuitive framework for real-time freeform modeling*, SIGGRAPH 04



Kobbelt, Campagna, Vorsatz, Seidel: *Interactive Multi-Resolution Modeling on Arbitrary Meshes*, SIGGRAPH 98

Outline

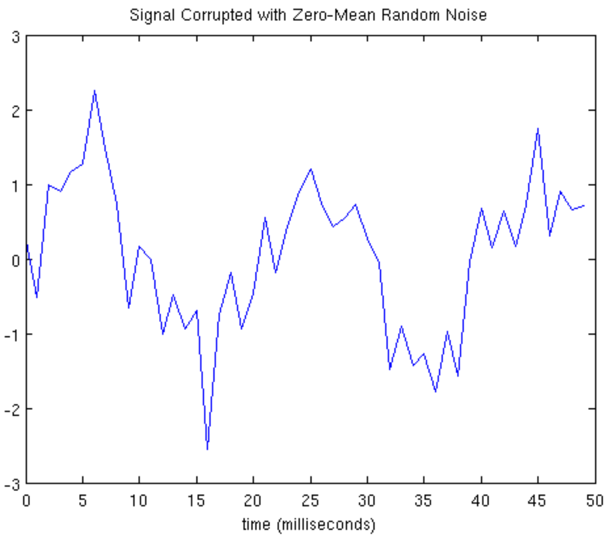
- Motivation
- Smoothing as Diffusion
- Smoothing as Energy Minimization
- Alternative Approaches

Outline

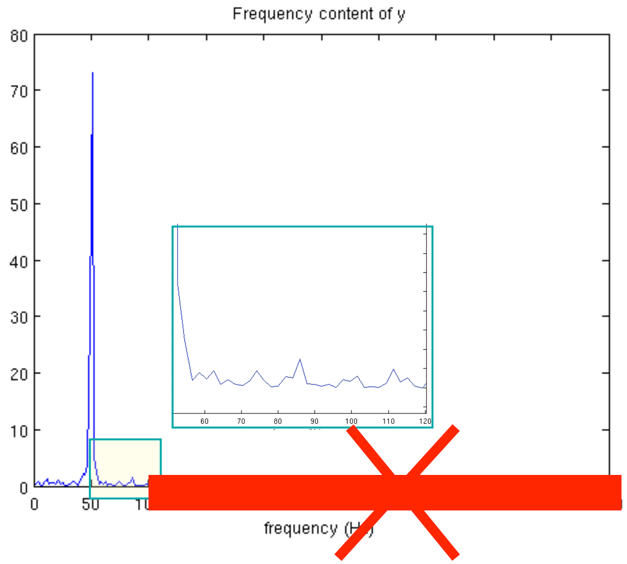
- Motivation
- Smoothing as Diffusion
  - **Spectral Analysis**
  - Laplacian Smoothing
  - Curvature Flow
- Smoothing as Energy Minimization
- Alternative Approaches

Filter Design

- Assume high frequency components = noise
- Low-pass filter



spatial domain

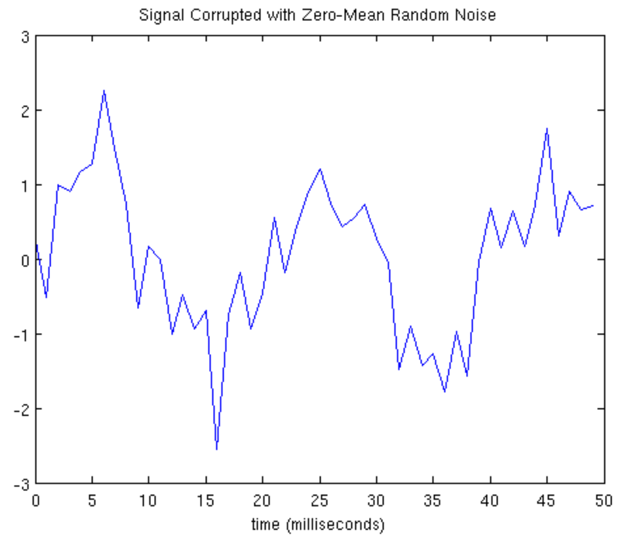


low pass

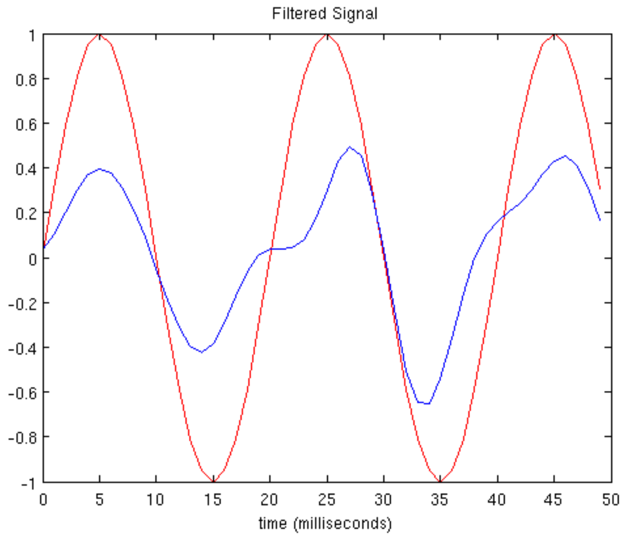
frequency domain

Filter Design

- Assume high frequency components = noise
- Low-pass filter



spatial domain



reconstruction = filtered signal

Filter Design

- Assume high frequency components = noise
- Low-pass filter
  - damps high frequencies (ideal: cut off)
  - e.g., by convolution with Gaussian (spatial domain)
    = multiply with Gaussian (frequency domain)
- Fourier Transform

Spectral Analysis and Filter Design

- Univariate: Fourier Analysis

$$F(\varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\varphi t} dt$$

frequency domain

spatial domain

- Example: Low-pass filter
  - Damp (ideally cut off high frequencies)
  - Multiply F with Gaussian (= convolve f with Gaussian)
- Are there "geometric frequencies"?

Spectral Analysis and Filter Design

- Univariate: Fourier Analysis

$$F(\varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\varphi t} dt$$

- Generalization

$$\Delta e^{i\varphi t} = \frac{\partial^2}{\partial t^2} e^{i\varphi t} = -\varphi^2 e^{i\varphi t}$$

  - $e^{i\varphi t}$ are eigenfunctions of the Laplacian
  - use them as basis functions for geometry

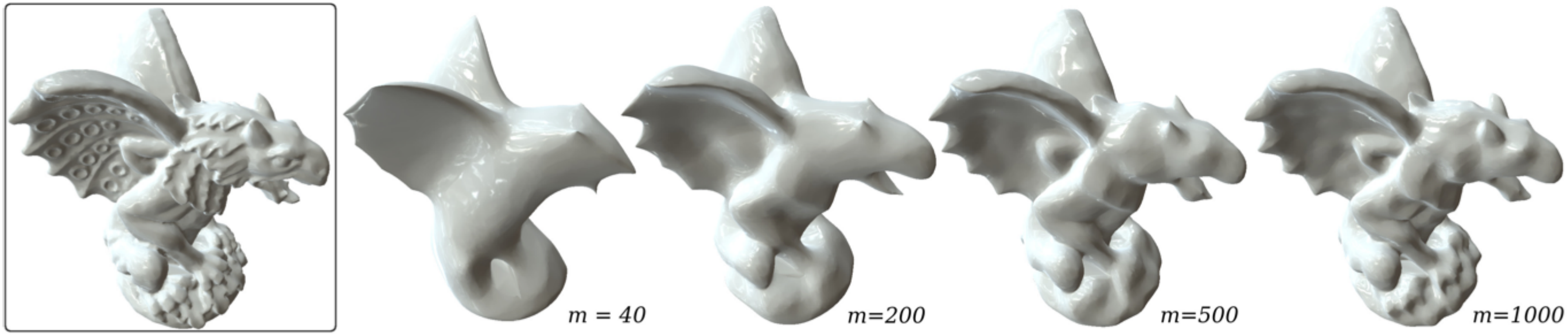
Spectral Analysis

- Eigenvalues of Laplacian ≅ frequencies

B. Vallet, B. Levy. *Spectral geometry processing with manifold harmonics*. Technical report, INRIA-ALICE, 2007.

Spectral Analysis

- Low-pass filter ≅ reconstruction from eigenvectors associated with low frequencies



B. Vallet, B. Levy. *Spectral geometry processing with manifold harmonics*. Technical report, INRIA-ALICE, 2007.

Spectral Analysis

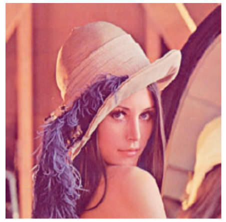
- Eigenvalues of Laplace matrix ≅ frequencies
- Low-pass filter ≅ reconstruction from eigenvectors associated with low frequencies
- Decomposition in frequency bands is used for mesh deformation
  - often too expensive for direct use in practice! difficult to compute eigenvalues efficiently
- For smoothing apply diffusion…

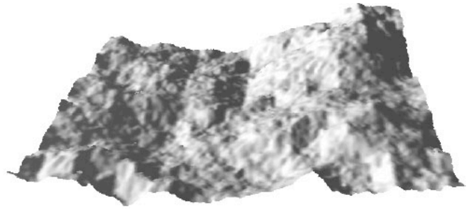
Outline

- Motivation
- Smoothing as Diffusion
  - Spectral Analysis
  - **Laplacian Smoothing**
  - Curvature Flow
- Smoothing as Energy Minimization
- Alternative Approaches

Diffusion

- Diffusion equation





diffusion constant

$$\frac{\partial}{\partial t} x = \mu \Delta x$$

Laplace operator

Diffusion

- Diffusion equation

diffusion constant

$$\frac{\partial}{\partial t} x = \mu \Delta x$$

Laplace operator

Laplacian Smoothing

- Discretization of diffusion equation

$$\frac{\partial}{\partial t}\mathbf{p}_i = \mu \Delta \mathbf{p}_i$$

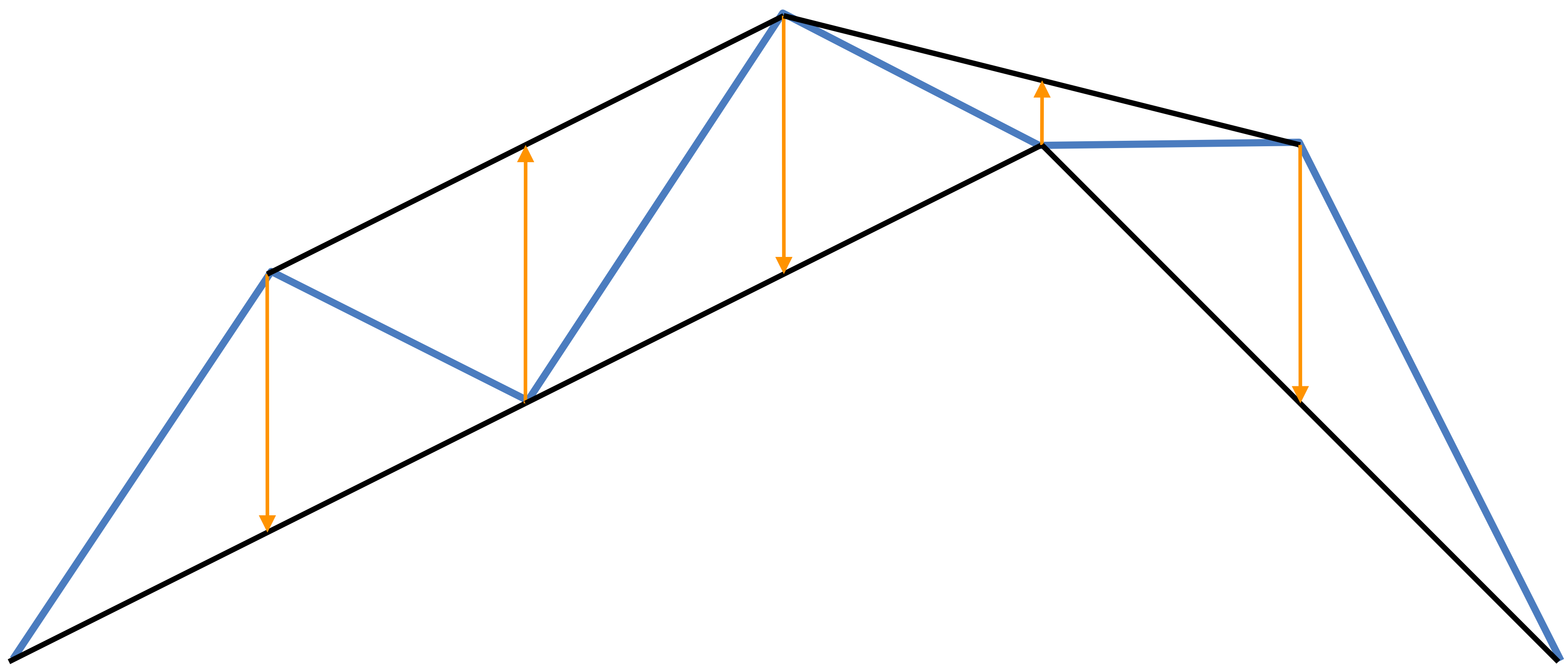
- Leads to simple update rule
  - iterate

$$\mathbf{p}_i' = \mathbf{p}_i + \mu \; dt \; \Delta \mathbf{p}_i$$

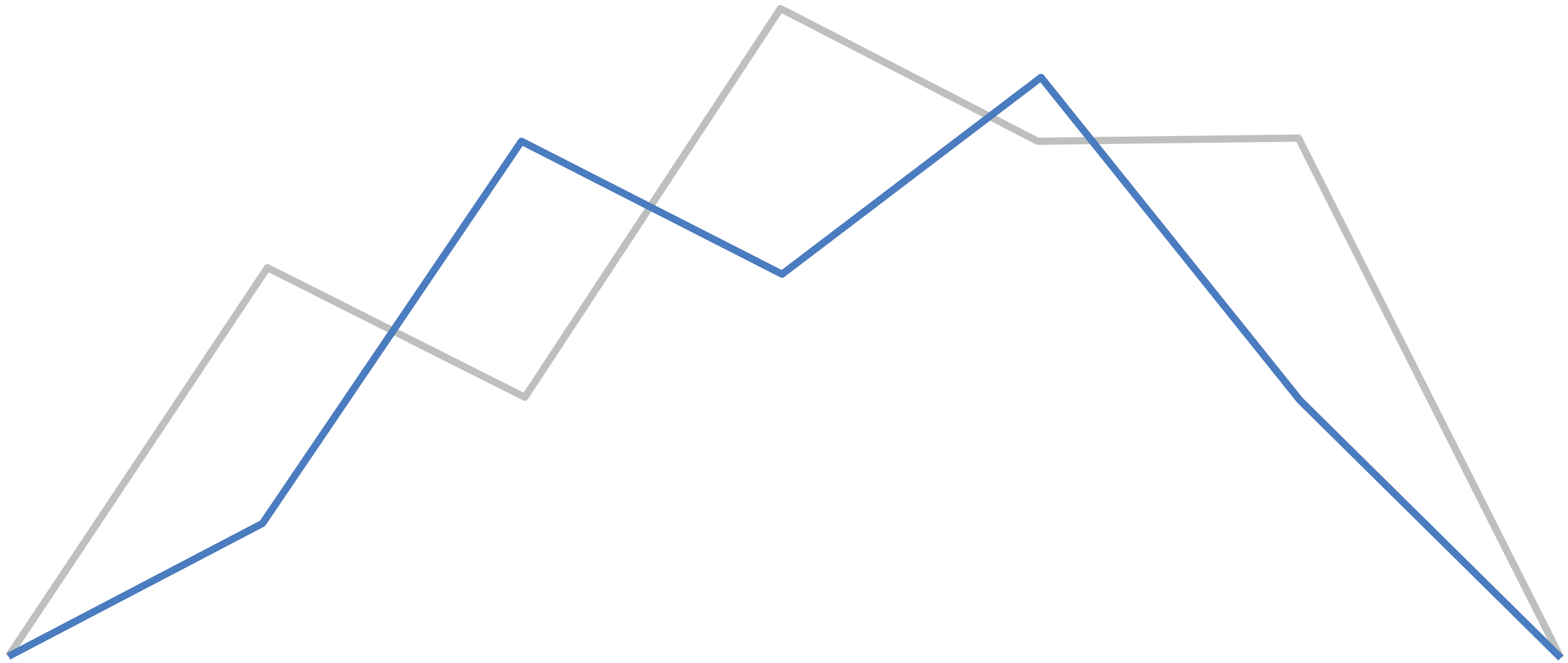
explicit Euler integration

  - until convergence

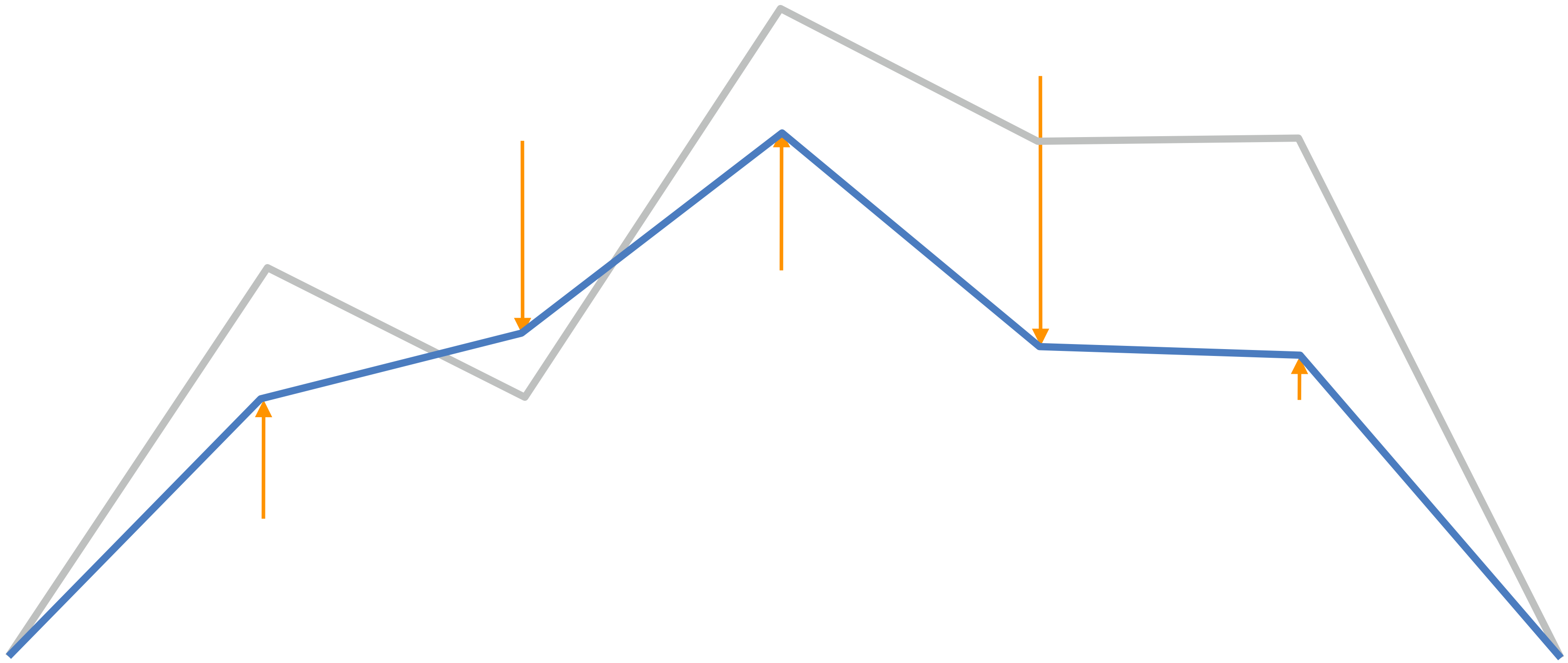
A Simple Example



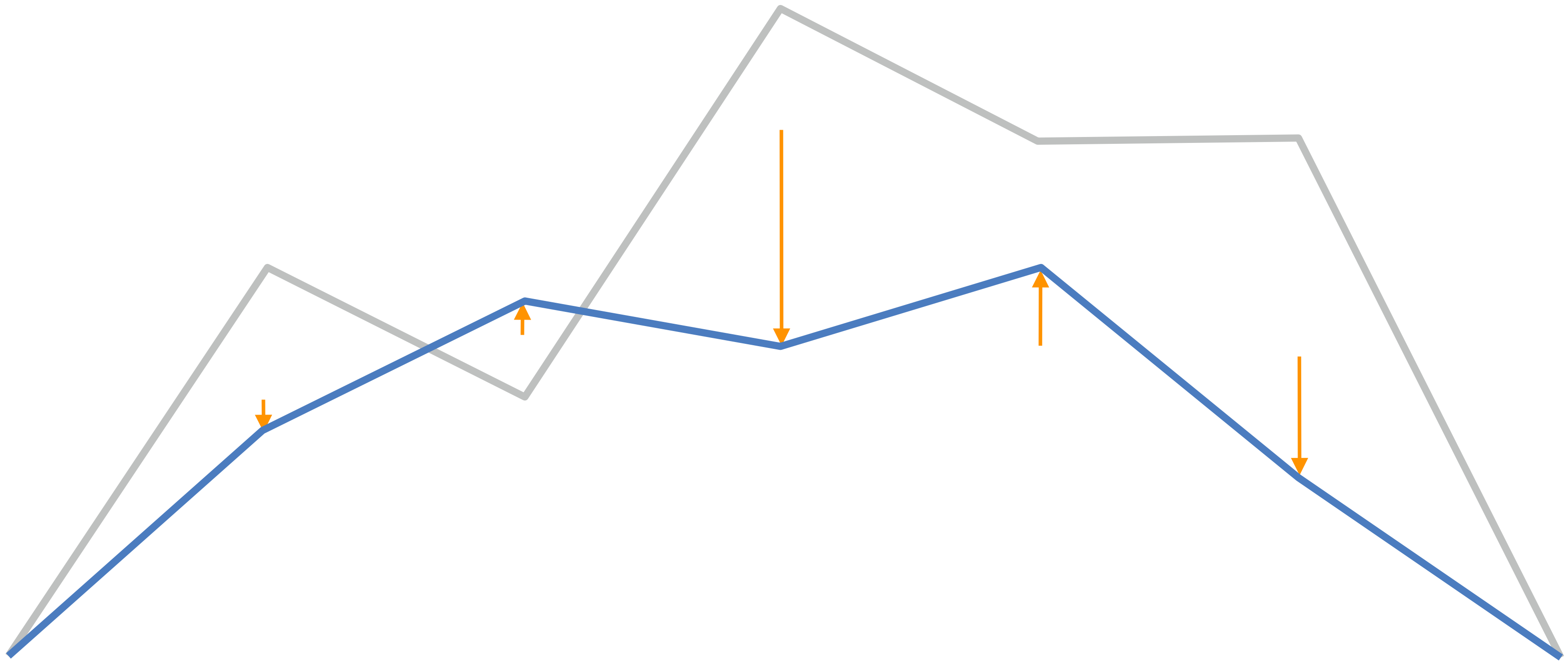
A Simple Example



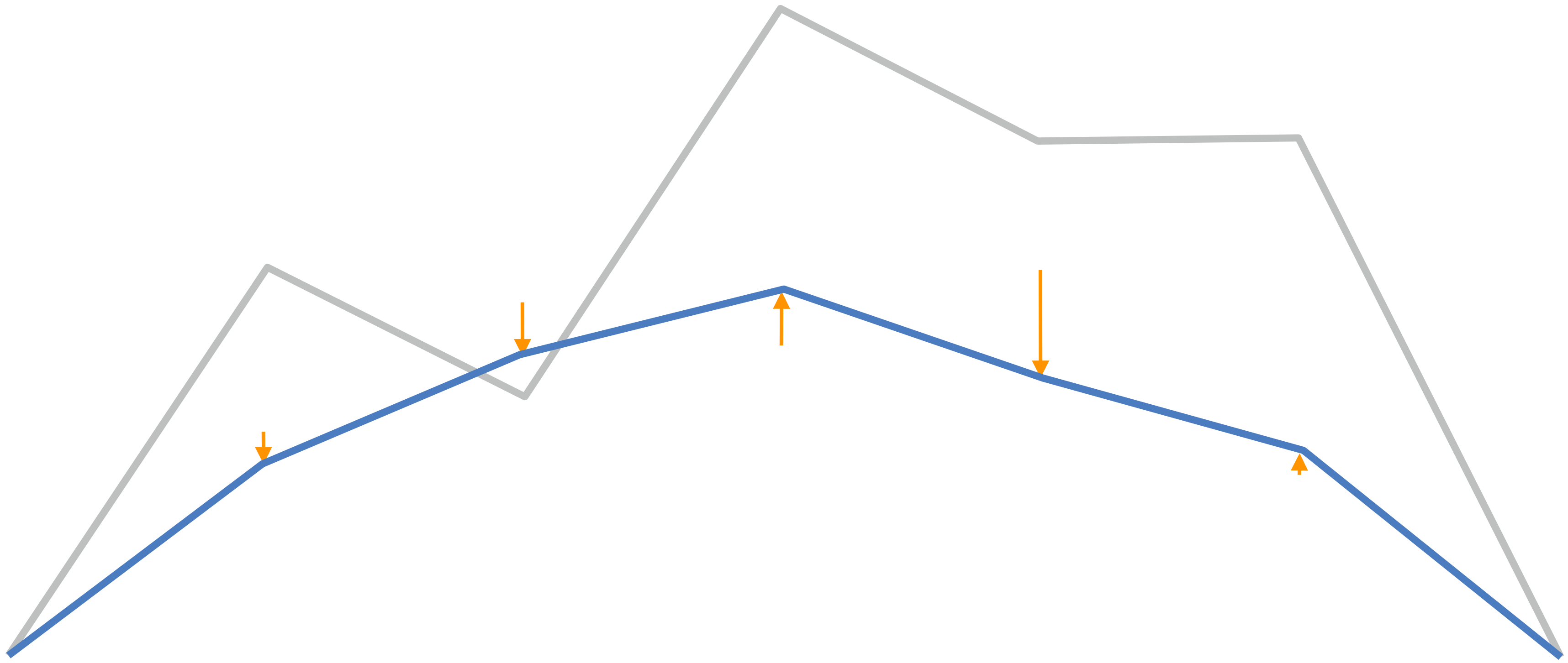
A Simple Example



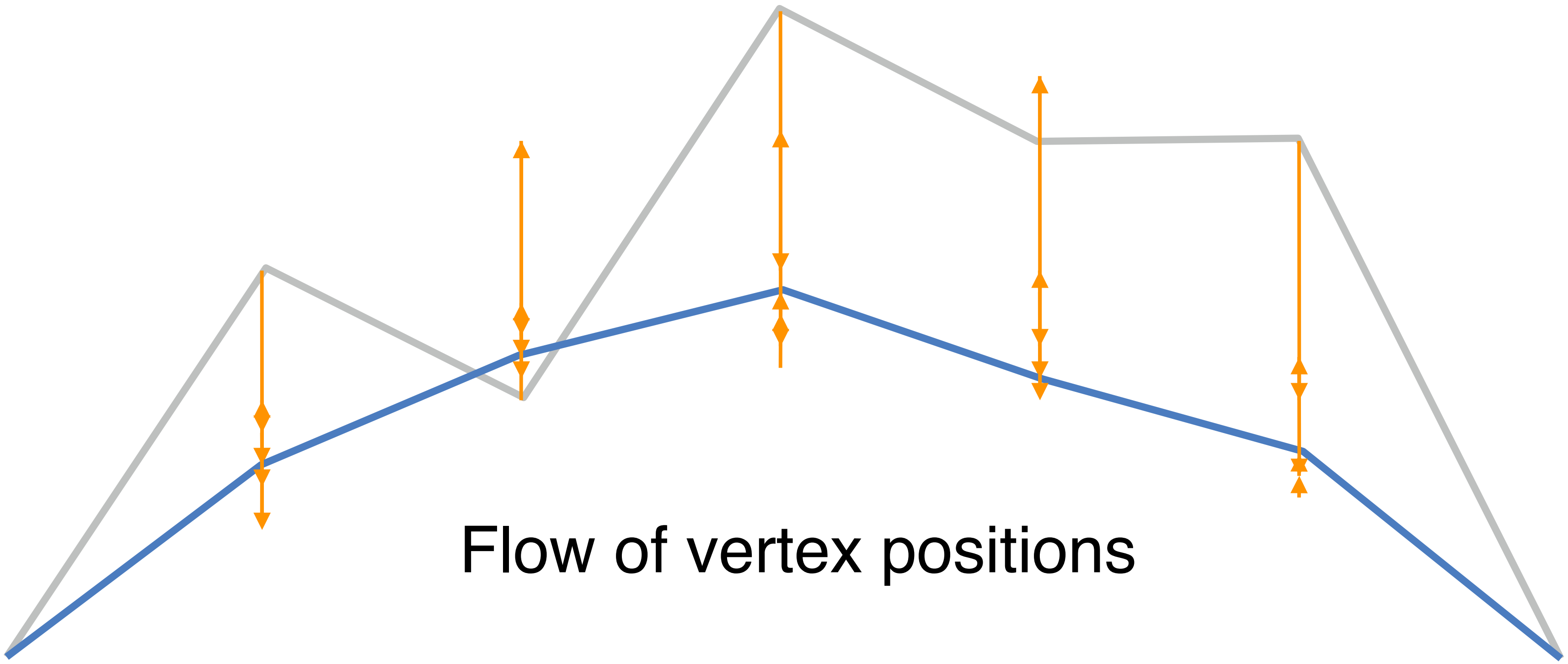
A Simple Example



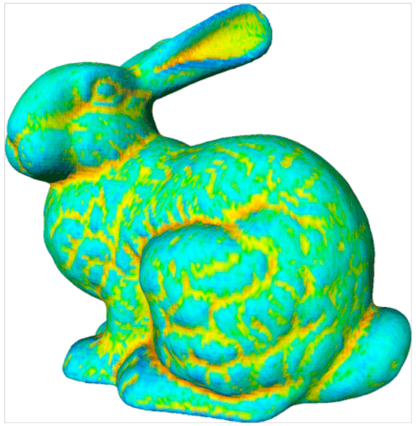
A Simple Example



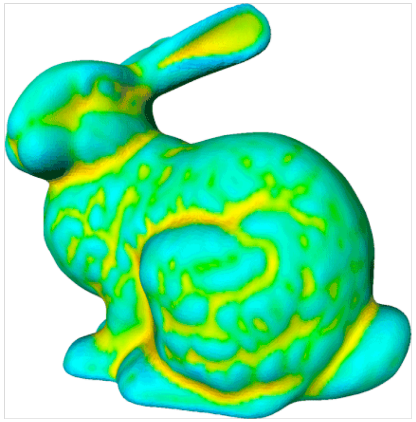
A Simple Example



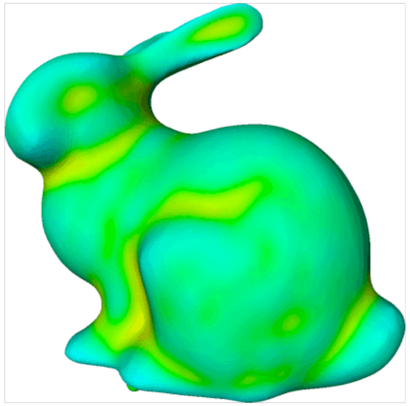
Laplacian Smoothing



0 Iterations



5 Iterations



20 Iterations

Outline

- Motivation
- Smoothing as Diffusion
  - Spectral Analysis
  - Laplacian Smoothing
  - **Curvature Flow**
- Smoothing as Energy Minimization
- Alternative Approaches

Curvature Flow

- Curvature is independent of parameterization
- Flow equation

surface normal

$$\frac{\partial}{\partial t}\mathbf{p} = -2\mu H \mathbf{n}$$

mean curvature

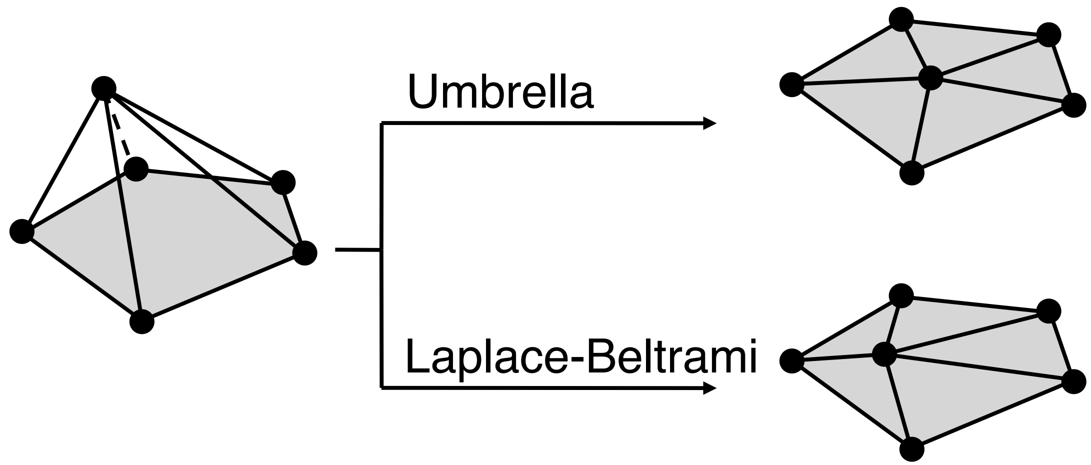
- We have

$$\Delta_S \mathbf{p} = -2H\mathbf{n}$$

Laplace-Beltrami operator

Curvature Flow

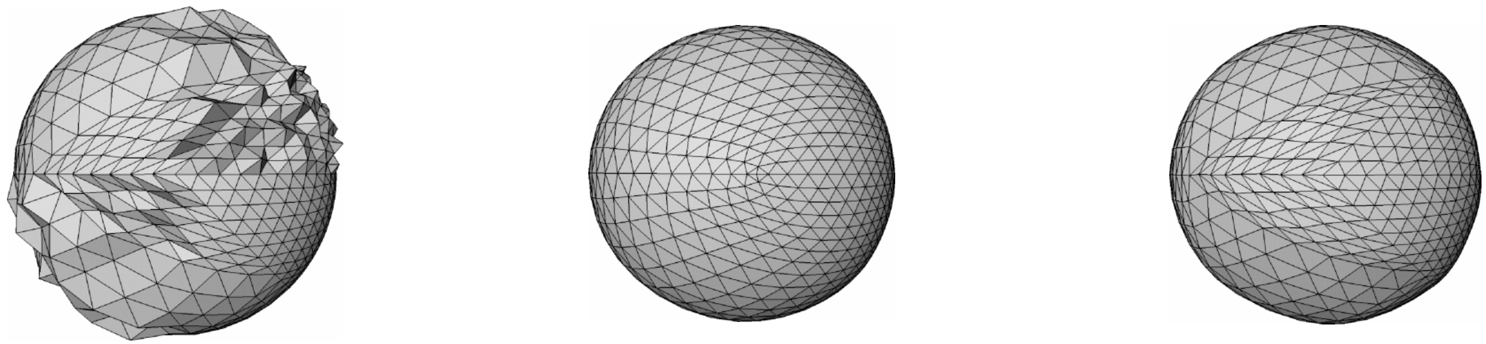
- Mean curvature flow $$\frac{\partial}{\partial t}\mathbf{p} = \mu \Delta_S \mathbf{p}$$
  - use discrete Laplace-Beltrami operator *(cot weights)*
- Compare to uniform discretization of Laplacian



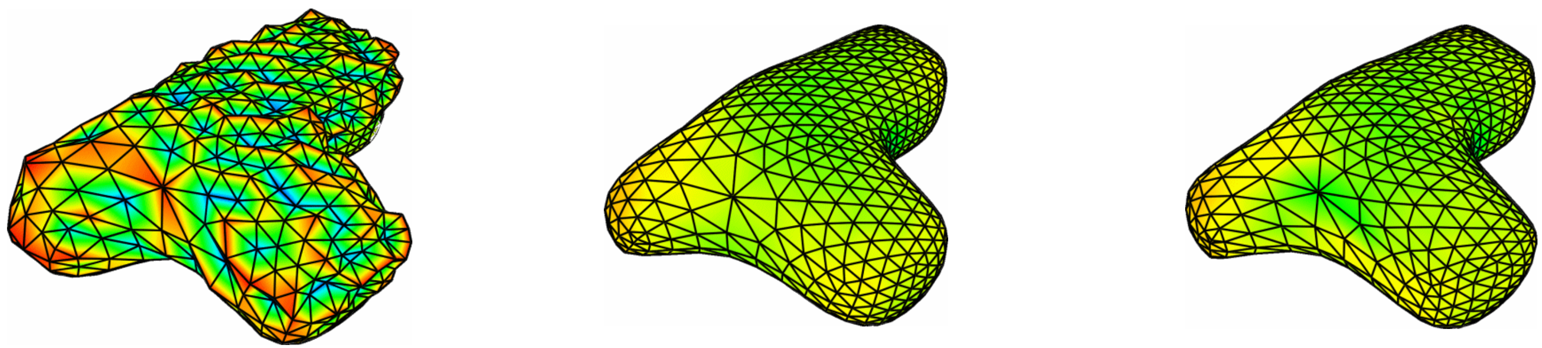
tangential drift

vertices move only along normal

Comparison



Original | Umbrella | Laplace-Beltrami



Outline

- Motivation
- Smoothing as Diffusion
- **Smoothing as Energy Minimization**
  - membrane energy
  - thin-plate energy
- Alternative Approaches

Energy Minimization

- Penalize "un-aesthetic behavior"
- Measure *fairness*
  - *principle of the simplest shape*
  - physical interpretation
- Minimize energy functional
  - examples: *membrane / thin plate energy*

Non-Linear Energies

- Membrane energy (surface area)

$$\int_{\mathcal{S}} \mathrm{d}s \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}$$

- Thin-plate surface (curvature)

$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \mathrm{d}s \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}, \;\; \mathbf{n}(\delta\mathcal{S}) = \mathbf{d}$$

- Too complex... simplify energies

Membrane Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\int_\Omega \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, \mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

Membrane Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\int_\Omega \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \,\mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

- Variational calculus

$$\Delta \mathbf{p} = 0$$

Thin-Plate Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$$

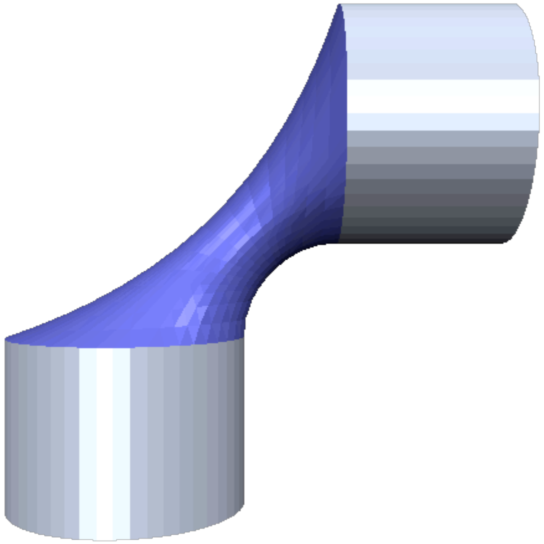
- Thin-plate energy (curvature)

$$\int_\Omega \|\mathbf{p}_{uu}\|^2 + 2\,\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \,\mathrm{d}u\mathrm{d}v \;\to\; \min$$

- Variational calculus

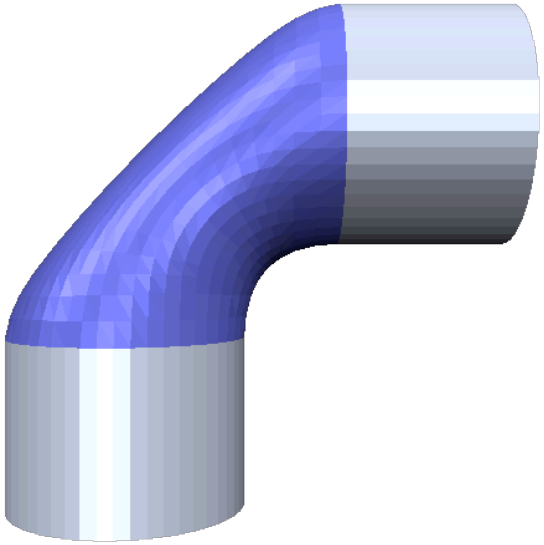
$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals



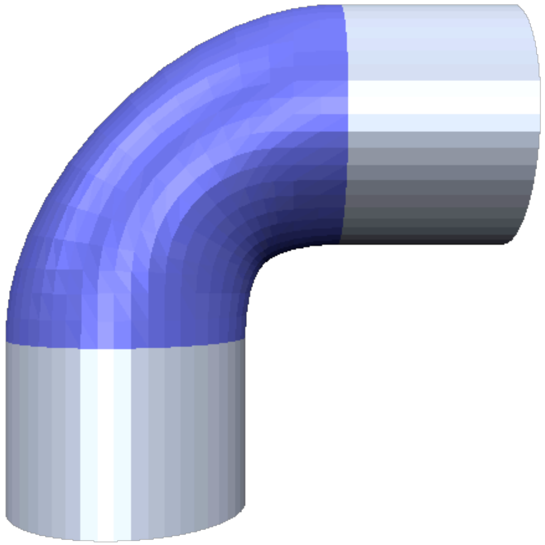
Membrane

$\Delta_{\mathcal{S}}\mathbf{p} = 0$



Thin Plate

$\Delta_{\mathcal{S}}^2\mathbf{p} = 0$



$\Delta_{\mathcal{S}}^3\mathbf{p} = 0$

Analysis

- Minimizer surfaces satisfy Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}^{k}\mathbf{p} = 0$$

- They are stationary surfaces of Laplacian flows

$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_{\mathcal{S}}^{k}\mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

Outline

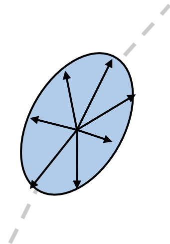
- Motivation
- Smoothing as Diffusion
- Smoothing as Energy Minimization
- **Alternative Approaches**

Alternative Approaches

- Anisotropic Diffusion
  - Data-dependent
  - Non-linear

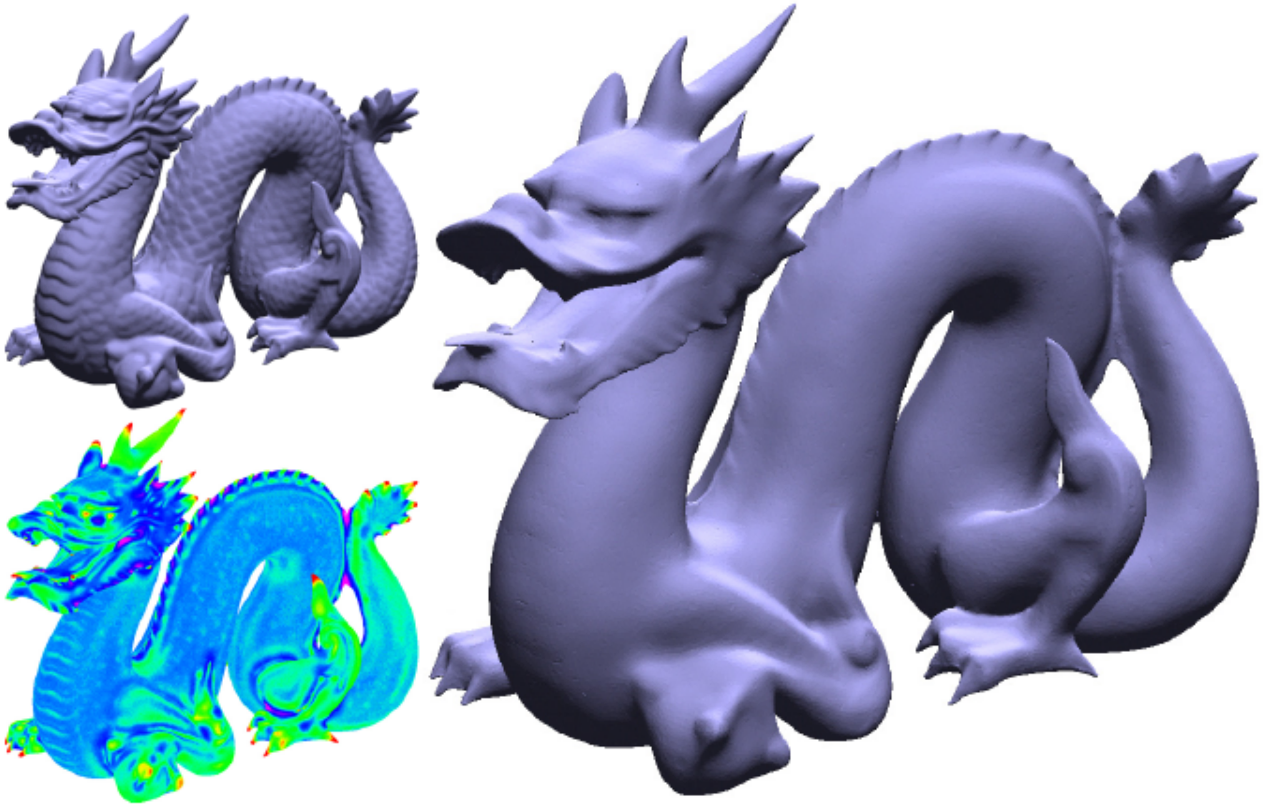
$$\frac{\partial}{\partial t}\mathbf{x} = \operatorname{div} D \, \nabla \mathbf{x}$$

diffusion tensor



- Normal filtering
  - Smooth normal field and reconstruct *(mesh editing)*
- Non-linear PDEs
  - Avoid parameter dependence for fair surface design
- Bilateral Filtering

Example of Bilateral Filtering



Jones, Durand, Desbrun: *Non-iterative feature preserving mesh smoothing*, SIGGRAPH 2003

Literature

- Taubin: *A signal processing approach to fair surface design*, SIGGRAPH 1996
- Desbrun, Meyer, Schroeder, Barr: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 99
- Botsch, Kobbelt: *An Intuitive Framework for Real-Time Freeform Modeling*, SIGGRAPH 2004
- Fleishman, Drori, Cohen-Or: *Bilateral mesh denoising*, SIGGRAPH 2003
- Jones, Durand, Desbrun: *Non-iterative feature preserving mesh smoothing*, SIGGRAPH 2003